On representable graphs

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ABSTRACT

A graph $G = (V, E)$ is representable if there exists a word W over the alphabet V such that letters x and y alternate in W if and only if $(x, y) \in E$ for each $x \neq y$. If W is k-uniform (each letter of W occurs exactly k times in it) then G is called k-representable. We prove that a graph is representable if and only if it is k-representable for some k. Examples of non-representable graphs are found in this paper. Some wide classes of graphs are proven to be 2- and 3-representable. Several open problems are stated.

Keywords: combinatorics on words, representation, (outer)planar graphs, prisms, Perkins semigroup, graph subdivisions

1. Introduction

To the nodes of a graph G we assign distinct letters from some alphabet. G is representable if there exists a word W such that any two letters, say x and y, alternate in W if and only if G contains an edge between the nodes corresponding to x and y. In such a situation we say that W represents G.

Representable (in our sense) graphs are considered in [1] to obtain asymptotic bounds on the free spectrum of the widely-studied *Perkins semigroup*, $\mathbf{B_2^1}$, which has played central role in semigroup theory since 1960, particularly as a source of examples and counterexamples. Recall that the Perkins semigroup has the elements

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and the operation is usual matrix-multiplication. A word W over the alphabet $X_n = \{x_1, \dots, x_n\}$ induces a function $f_W : (\mathbf{B_2^1})^n \to \mathbf{B_2^1}$ as follows: for $(a_1, \dots, a_n) \in (\mathbf{B_2^1})^n$,

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let $f_W(a_1, \ldots, a_n)$ be the evaluation of W after substituting a_i wherever x_i occurs for $1 \leq i \leq n$. Given n, how many distinct functions can be represented by words in the alphabet X_n? We refer to [3] for a related problem as well as for some references in the subject. It turns out ([1]) that the last question is equivalent to finding the number of representable graphs on n nodes.

Our studies are mainly motivated by the fact that it is unknown how many graphs can be represented, and for some graphs known to be representable, explicit constructions of "words-representants" are missing. In this paper our ultimate goal is not counting or improving asymptotics for representable graphs, but rather enlarging the set of known classes of representable graphs by providing explicit constructions of words representing them, which is often a challenging combinatorics on words problem. Our results give the structure of words-representants in many cases. Sometimes, we even study different ways to represent a graph: we deal with k-representations — the multiplicity of each letter must be k in the words. We believe that our results could be useful in considering the general problem (the classifying all the graphs by the property "to be representable"), which in turn would improve a lower bound for the number of representable functions for the Perkins semigroup.

The approach in [1] is not the first instance when combinatorics on words is used to solve an algebraic problem. A classical example is the following Burnside-type problem: The element z is a zero element of a semigroup S with an associative operation $\cdot$, if $z \cdot a = a \cdot z = z$ for all a in S; Let S be a semigroup generated by three elements, such that the square of every element in S is zero (thus $a \cdot a = z$ for all a in S). Does S have an infinite number of elements? This question was answered in affirmative independently by Thue (1906), by Arshon (1937), and by Morse (1938). All of the solutions used combinatorics on words approaches based on the morphism-type constructions of infinite square-free sequences. To find out more on applications of combinatorics on words methods for solving problems arising in algebra, theoretical computer science, dynamical system, number theory and other areas we refer to [2]. So, beyond applications to the problem on the Perkins semigroup, our paper has independent interest from a pure combinatorics on words point of view in form of constructions we use to represent graphs.

The paper is organized as follows. In Section 2 we give formal definitions and state some general properties of representable graphs. In Section 3 examples of non-representable graphs are presented. In Section 4 the class of 2-representable graphs is studied. In particular, it is proved that all outerplanar graphs are 2-representable. In Section 5 we prove that 3-subdivision of every graph is 3-representable (hence, every graph can be a minor of a 3-representable graph). Section 6 is devoted to some open problems.

2. Preliminaries

In this section we give formal definitions of studied objects and make some simple observations on their properties.

Let W be a finite word over an alphabet $\{x_1, x_2, \ldots\}$. If W involves the variables $x_1, x_2, \ldots, x_n$ then we write $Var(W) = \{x_1, \ldots, x_n\}$. Let X be a subset of $Var(W)$.

Then $W \setminus X$ is the word obtained by eliminating all variables in X from W. A word is k-*uniform* if each letter appears in it exactly k times. A 1-uniform word is also called a *permutation*. Denote by W_1W_2 the *concatenation* of the words W_1 and W_2. We say that the letters x_i and x_j *alternate* in W if a subword induced by these two letters contains neither x_ix_i nor x_jx_j as a factor. For example, x_1 and x_2 alternate in the word $x_1x_2x_3x_4x_3x_1x_4x_2x_1x_3x_2x_4x_1$ while x_2 and x_4 do not alternate there. If a word W contains k copies of a letter x then we denote these k appearances of x by $x^1, x^2, \ldots, x^k$. We write $x_i^j < x_k^l$ if x_i^j stays in W before x_k^l, i. e. x_i^j is to the left of x_k^l in W.

Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. We say that a word W *represents* the graph G if there is a bijection $\phi : Var(W) \to V$ such that $(\phi(x_i), \phi(x_j)) \in E$ if and only if x_i and x_j alternate in W. It is convenient to identify the vertices of a representable graph and the corresponding letters of a word representing it. We call a graph G *representable* if there exists a word W that represents G. We denote the set of all words representing G by $\mathcal{R}(G)$. If G can be represented by a k-uniform word, then we say that G is k-*representable* and $\mathcal{R}_k(G)$ denotes the set of all k-uniform words that represent G. Clearly, the complete graphs are the only examples of 1-representable graphs. So, in what follows we assume that $k \geq 2$.

Observation 1 *If $W \in \mathcal{R}(G)$ then its reverse W^{-1} (the word W written in reverse order) also represents G, that is, $W^{-1} \in \mathcal{R}(G)$.*

If W represents G and $X \subset Var(W)$ then clearly $W \setminus X$ represents $G \setminus X$ — the subgraph of G induced by the vertices from $V(G) \setminus X$. So, we can make the following

Observation 2 *The class of (k-)representable graphs is hereditary, i.e., if G is (k-) representable then all its induced subgraphs are (k-)representable.*

The next observation follows directly from the definition of representable graphs, but it helps much to verify whether a given word represents a desired graph or not.

Observation 3 *Let $W = W_1x^iW_2x^{i+1}W_3$ be a word representing G where x^i and x^{i+1} are two consecutive occurrences of a letter x in W. Let X be the set of all letters that appear only once in W_2. Then the vertex x is not adjacent to $Var(W) \setminus X$ in G, i. e., all possible candidates for x to be adjacent to in G are in X.*

Suppose $P(W)$ is the permutation obtained by removing all but the leftmost occurrences of a letter x in W for each $x \in Var(W)$. We call $P(W)$ the *initial permutation* of W.

Observation 4 *Let $W \in \mathcal{R}(G)$ and $P(W)$ be its initial permutation. Then $P(W)W \in \mathcal{R}(G)$. In particular, for every $k > l$, an l-representable graph is also k-representable.*

Indeed, if x and y do not alternate in W then they also do not alternate in $P(W)W$. Otherwise, the order of x and y in the permutation $P(W)$ is the same as that in W, and thus x and y also alternate in $P(W)W$.

Note that Observation 4 shows that any word W that represents a graph G can always be extended to the left to a word representing G. It is clear how to use the same idea to make an extension of W to the right to a word representing G (one basically needs to replace "leftmost" by "rightmost" in the definition of the initial permutation).

It follows from Observations 2 and 4 that a graph $G \cup H$ (G and H are two disjoint connected components of the graph) is representable if and only if G and H are representable (just take concatenation of a word in $\mathcal{R}(G)$ and a word in $\mathcal{R}(H)$ both having at least two occurrences of each letter). So, we may consider only connected graphs.

Now let us prove some properties of k-representable graphs.

Proposition 5 *Let $W = AB$ be a k-uniform word representing a graph G, that is, $W \in \mathcal{R}_k(G)$. Then the word $W' = BA$ also k-represents G.*

Proof. The claim follows from the fact that under a cyclic shift of B in W, the subword $xyxy \dots xy$ can be transformed either to the same subword or to the subword $yxyx \dots yx$, and no other subword consisting of k copies of x and y can be transformed to these subwords. In other words, x and y alternate in W if and only if they alternate in W'. □

Proposition 6 *Let W_1 and W_2 be k-uniform words ($k \geq 2$) representing graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively where V_1 and V_2 are disjoint. Suppose that $x \in V_1$ and $y \in V_2$. Let H_1 be the graph $(V_1 \cup V_2, E_1 \cup E_2 \cup \{(x, y)\})$. Also denote by H_2 the graph obtained from G_1 and G_2 by identifying x and y into a new vertex z. Then both H_1 and H_2 are k-representable.*

Proof. By Proposition 5, we may assume that $W_1 = A_1x^1A_2x^2 \dots A_kx^k$ and $W_2 = y^1B_1y^2B_2 \dots y^kB_k$ where $A_1A_2 \dots A_k = W_1 \setminus \{x\}$ and $B_1B_2 \dots B_k = W_2 \setminus \{y\}$. Then the words

$$W_3 = A_1x^1A_2y^1x^2B_1A_3 \dots y^{k-2}x^{k-1}B_{k-2}A_ky^{k-1}x^kB_{k-1}y^kB_k$$

and

$$W_4 = A_1z^1A_2B_1z^2A_3B_2z^3 \dots A_kB_{k-1}z^kB_k$$

represent H_1 and H_2 respectively. We will prove it for W_3 only, since the proof for W_4 is analogous (and somewhat easier).

Clearly, x and y alternate in W_3, so, they are adjacent in H_1. Note, that $W_3 \setminus Var(W_2) = W_1$ and $W_3 \setminus Var(W_1) = W_2$. So, the graph induced by $Var(W_i)$ is isomorphic to G_i for $i = 1, 2$ and it is enough to show that there are no edges between them except for (x, y). Let $a \in V_1, b \in V_2$, and $\{a, b\} \neq \{x, y\}$, say, $a \neq x$. Since W_1 is k-uniform, there are k occurrences of a in W_1. Then among $k-1$ subsets $A_1 \cup A_2, A_3, A_4, \dots, A_k$ there exists one containing at least two copies of a. Hence, the subword induced by a and b contains aa as a factor, and $(a, b) \notin E(H_1)$. □

We conclude the section by a theorem showing significance of k-representable graphs in representing graphs theory.

Theorem 7 *A graph G is representable if and only if there is k such that G is k-representable.*

Proof. Clearly if G is k-representable then it is representable.

Conversely, suppose a word W represents G and the maximum number of copies of a letter in W is k. To avoid triviality, we assume that W is not k-uniform. Let S be obtained from W by removing all the copies of the letters appearing in W k times. Then the word $P(S)W$, where $P(S)$ is the initial permutation of S (see definition right above Observation 4), also represents G. Indeed, if (x, y) is not an edge in G then x and y do not alternate in W and thus they do not alternate in $P(S)W$. On the other hand, if (x, y) is an edge in G (x and y alternate in W), the following (non-equivalent) cases are possible:

- $x \notin S$ and $y \notin S$: clearly x and y alternate in $P(S)W$;
- $x \in S$ and $y \in S$: by definition, the order of x and y in the permutation $P(S)$ is the same as that in W, and thus x and y alternate in $P(S)W$;
- $x \in S$ and $y \notin S$: from definition of S, y must occur k times in W. But then x must occur $k-1$ times in W and the alternating subword of W induced by x and y must begin and end by y. This shows that x and y alternate in $P(S)W$.

So, the word $P(S)W$ represents G. The length of $P(S)W$ is larger than the length of W (W is not k-uniform), and the maximum number of occurrences of a letter in $P(S)W$ is still k. Thus, we can repeat the procedure above, if necessary, until we get a k-uniform word representing G. □

3. Non-representable graphs

Are there any non-representable graphs? In this section we give the positive answer to this question.

We call a graph *permutationally representable* if it can be represented by a word of the form $P_1P_2\dots P_k$ where all P_i are permutations. In particular, all permutationally representable graphs are k-representable. Such graphs were studied in [1] where the following statement was proved:

Lemma 8 *A graph is permutationally representable if and only if at least one of its possible orientations is a comparability graph of a poset. In particular, all bipartite graphs are permutationally representable.*

Recall that a directed graph is called a comparability graph of a poset if the relation $(x, y) \in E$ is transitive, i. e., whenever $(x, y) \in E$ and $(y, z) \in E$ there must be $(x, z) \in E$.

The following lemma provides a relation between permutationally representable and representable graphs.

Lemma 9 *Let $x \in V(G)$ be a vertex of degree $n-1$ in G where $n = |V|$. Let $H = G \setminus \{x\}$. Then G is representable if and only if H is permutationally representable.*

Proof. If H can be represented by a word $W = P_1P_2 \dots P_k$ where all P_i are permutations then the word $P_1x^1P_2x^2 \dots P_kx^k$ clearly represents G.

If G is representable, then by Theorem 7 and Proposition 5 it can be represented by a k-uniform word $W = x^1P_1x^2P_2 \dots P_{k-1}x^kP_k$. Then by Observation 3 each P_i for $i = 1, 2, \dots, k-1$ must be a permutation. Since W is k-uniform, P_k is also a permutation. Clearly, the word $P_1P_2 \dots P_k$ represents H. So, H is permutationally representable. □

Lemmas 8 and 9 give us the method to construct non-representable graphs. We may take a graph having no orientation which is a comparability graph of a poset (the smallest one is C_5), and add an all-adjacent vertex to it. In particular, all odd wheels W_{2t+1} for $t \geq 2$ are non-representable graphs. Some examples of small non-representable graphs can be found in Figure 1.

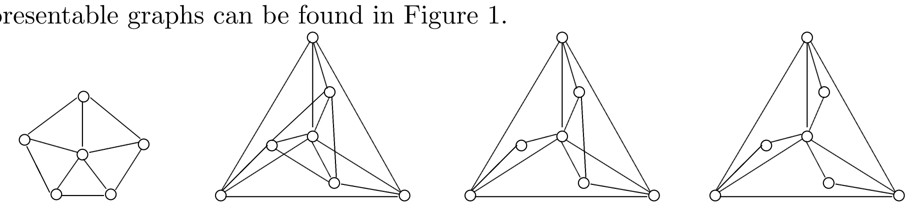

Fig. 1. Small non-representable graphs

For a vertex $x \in V(G)$ denote by $N(x)$ the set of all its neighbors. By Observation 2 and Lemma 9, we have the following

Theorem 10 *If G is representable then for every $x \in V(G)$ the graph induced by $N(x)$ is permutationally representable.*

Unfortunately, we have no examples of non-representable graphs, that does not satisfy the conditions of Theorem 10.

In the next two sections we present some methods to construct representable graphs.

4. 2-representable graphs

A nice property of 2-representable graphs is that the necessary condition of Observation 3 is sufficient for them. This means that a vertex x of a 2-representable graph is adjacent to those and only those vertices, whose letters appear exactly once between x^1 and x^2 in the word $W(G)$.

A graph G is *outerplanar* if it can be drawn at the plane in such a way that no two edges meet in a point other than a common vertex and all the vertices lie in the outer face. Since all wheels are planar, there are planar non-representable graphs. However, all outerplanar graphs are 2-representable. We prove even a bit stronger result.

Theorem 11 *If a graph G is outerplanar then it is 2-representable. Moreover, if G is also 2-connected then it can be represented by such a word W that every edge (x, y) of the outer face appears as a factor (xy or yx) in W and these factors do not overlap for different edges of the outer face.*

Proof. We prove the theorem by induction on n, the number of vertices. For $n = 1$, the statement holds, since the only node, say x, gives the word xx representing a graph without edges.

If G has cut-vertices then we apply induction to its blocks and then connect them together using the technique of Proposition 6. Thus, it is enough to prove the second part of the theorem, i. e., the case when G is a 2-connected outerplanar graph.

Let $x_1x_2 \dots x_n$ be the outer face of G. If G has no chords, that is, G is a cycle, then it is easy to check by Observation 3 that the 2-uniform word

$$W = x_2x_1x_3x_2x_4x_3 \dots x_nx_{n-1}x_1x_n$$

represents G and satisfies the second condition of the theorem. In particular, $n = 3$ provides the induction basis for the second part of the theorem.

Suppose now that G has a chord (x_i, x_j) where $i < j - 1$. Consider two outerplanar 2-connected graphs G_1 and G_2 with the outer faces $x_1x_2 \dots x_ix_jx_{j+1} \dots x_n$ and $x_ix_{i+1} \dots x_j$ respectively. By induction, both of them are 2-representable. Moreover, using the induction hypothesis for second condition of the theorem, Proposition 5 and, if necessary, Observation 1, we can assume that the words representing G_1 and G_2 have form $W(G_1) = W_1x_ix_j$ and $W(G_2) = x_ix_jW_2$. Moreover, these words contain non-overlapping factors for all of the edges of the outer faces. But then the word $W = W_1W_2$ represents G and satisfies the second condition of the theorem. □

Note, that the class of 2-representable graphs is wider than the class of the outerplanar graphs. For example, graphs K_n and $K_{2,n}$ are 2-representable for every n. However there are graphs that are representable, but not 2-representable.

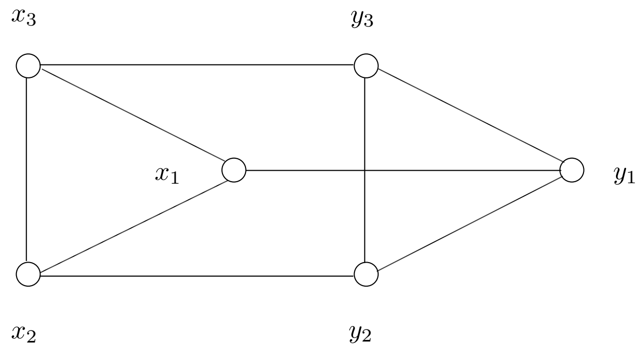

Fig. 2. Triangular prism is representable but not 2-representable graph

All graphs on $n \leq 5$ vertices are 2-representable. Indeed, by Proposition 6 it is enough to prove this only for 2-connected graphs. There are 14 2-connected graphs on at most 5 vertices, and 6 of them are outerplanar. Complete graphs K_4 and K_5 are 1-representable. The 2-uniform words representing the remaining 6 graphs are given below:

$x_1x_2x_3x_4x_1x_5x_2x_3x_4x_5$; $x_1x_2x_3x_1x_4x_5x_2x_3x_4x_5$; $x_1x_2x_3x_4x_1x_5x_2x_4x_3x_5$;

$x_1x_2x_3x_1x_4x_5x_3x_2x_4x_5$; $x_1x_2x_3x_1x_4x_5x_2x_3x_5x_4$; $x_1x_2x_3x_1x_4x_5x_3x_2x_5x_4$.

There are 56 2-connected graphs on $n = 6$ vertices; 54 of them are 2-representable (the authors have a list of the corresponding 2-uniform words). By Theorem 10, the wheel W_5 is not representable at all. Finally, the triangular prism (see Figure 2) is the smallest example of representable but not 2-representable graph, as follows from Propositions 12 and 15 below.

Proposition 12 *The triangular prism G in Figure 2 is not 2-representable.*

Proof. Suppose that G is 2-representable. Denote by x such a letter that no letter occurs twice between x^1 and x^2. By Observation 3, there are only three neighbors of x between x^1 and x^2. Due to the symmetry and Proposition 5, we may assume that the word starts with x and $x = x_1$. By Observation 1 and the symmetry of the vertices x_2 and x_3, there are two non-equivalent cases.

1) The word representing G starts with $x_1^1x_2^1y_1^1x_3^1x_1^2$. Then $y_1^2 < x_2^2, y_1^2 > x_3^2$, and $x_2^2 < x_3^2$ since $(x_2, y_1) \notin E, (x_3, y_1) \notin E$, and $(x_2, x_3) \in E$. But it is clearly impossible.

2) The word representing G starts with $x_1^1x_2^1x_3^1y_1^1x_1^2$. Then $y_1^2 < x_2^2 < x_3^2$. Since y_3 is adjacent both to y_1 and x_3, there must be $y_3^1 < y_1^2$ and $y_3^2 > x_3^2$. But then x_2 and y_3 alternate, a contradiction with $(x_2, y_3) \notin E$. □

We will show in the next section that all prisms are 3-representable.

5. 3-representable graphs

Our main tool for constructing 3-representable graphs is the following

Lemma 13 *Let G be a 3-representable graph and $x, y \in V(G)$. Denote by H the graph obtained from G by adding to it a path of length at least 3 connecting x and y. Then H is also 3-representable.*

Proof. By Proposition 6 we can always add a leaf (a vertex of degree 1) to any place of a graph. Therefore, it is enough to prove the lemma for a path of length 3. In this case $V(H) = V(G) \cup \{u, v\}$ and $E(H) = E(G) \cup \{(x, u), (u, v), (v, y)\}$. Let W be a 3-uniform word representing G. We may assume that $x^1 < y^1$. Five cases are possible. We consider only one of them in details, since the proof is similar in all the cases.

1) $y^2 < x^2$ and $y^3 < x^3$. Then we substitute y^2 by $v^1u^1y^2v^2$ and x^3 by $u^2x^3v^3u^3$. We get a 3-uniform word W'. Since $W' \setminus \{u, v\} = W$, we need only to check the neighborhoods of u and v. Clearly, these vertices are adjacent. By Observation 3, u could also be adjacent only to x and v — only to y. The edges (u, x) and (v, y) indeed exist because of the corresponding subwords $x^1u^1x^2u^2x^3u^3$ and $y^1v^1y^2v^2y^3v^3$ in W'.

2) $y^2 < x^2$ but $x^3 < y^3$. Then we substitute y^1 by $v^1u^1y^1v^2$ and x^3 by $u^2x^3v^3u^3$.

3) $x^2 < y^2 < x^3$. Then we substitute x^1 by $u^1v^1x^1u^2$ and y^2 by $v^2y^2u^3v^3$.

4) $x^3 < y^2$ and $x^2 < y^1$. Then we substitute x^2 by $u^1x^2v^1u^2$ and y^2 by $v^2u^3y^2v^3$.

5) $x^3 < y^2$ but $y^1 < x^2$. Then we substitute x^2 by $u^1x^2v^1u^2$ and y^3 by $v^2u^3y^3v^3$. □

Note that the rightmost example in Figure 1 shows that it is not possible to reduce the length of path from 3 to 2 in Lemma 13.

A *subdivision* of G is a graph obtained from G by substitution of all of its edges by simple paths. A subdivision is called a *k-subdivision* if each of these paths has length at least k. The next theorem follows immediately from Lemma 13.

Theorem 14 *For every graph G there exists a 3-representable graph H that contains G as a minor. In particular, a 3-subdivision of every graph G is 3-representable.*

Another nice class of 3-representable graphs is the class of prisms (a prism is a graph consisting of two cycles $x_1x_2 \ldots x_t$ and $y_1y_2 \ldots y_t$ joined by the edges (x_i, y_i), $i = 1, 2, \ldots, t$; in particular, the 3-dimensional cube is a prism).

Proposition 15 *Every prism is 3-representable.*

Proof. We start with the following 3-uniform word

$$W_2 = x_1x_2y_1x_1y_2x_2y_1y_2x_1y_1x_2y_2$$

representing the 4-cycle $x_1x_2y_2y_1$. Note that W_2 contains $x_1^1x_2^1$ and $y_1^2y_2^2$ as factors. Add the path $x_2x_3y_3y_2$ to it using the third rule of Lemma 13 to get the 3-uniform word

$$W_3 = x_1x_3y_3x_2x_3y_1x_1y_2x_2y_1y_3y_2x_3y_3x_1y_1x_2y_2$$

that satisfies the following properties for $i = 3$:

1) W_i contains $x_1^1 x_i^1$ and $y_1^2 y_i^2$ as factors;

2) The subword of W_i induced by x_1 and x_i is $x_1^1 x_i^1 x_i^2 x_1^2 x_i^3 x_1^3$, while the subword induced by y_1 and y_i has the form $y_i^1 y_1^1 y_1^2 y_i^2 y_i^3 y_1^3$.

Repeat the operation of adding path $x_i x_{i+1} y_{i+1} y_i$ for $i = 3, 4, \ldots, t-1$. Since $(x_i, y_i) \in E$ we may always do it using the third rule of Lemma 13. It is easy to see that properties 1) and 2) hold for every i. The word W_t represents a prism without the edges (x_1, x_t) and (y_1, y_t). Now substitute factors $x_1^1 x_t^1$ and $y_1^2 y_t^2$ in W_t by $x_t^1 x_1^1$ and $y_t^2 y_1^2$, respectively. The word obtained represents the prism. Indeed, due to property 2), (x_1, x_t) and (y_1, y_t) become edges, and all the other adjacencies in the graph are not changed, since the subwords induced by any other pair of letters remain the same. □

6. Open problems

There are several problems that still remain unsolved.

First of all, we are interested in constructing new examples of non-representable graphs. Note, that since all bipartite graphs are permutationally representable, the chromatic number of all examples satisfying the conditions of Theorem 10 is at least 4.

Problem 1 *Are there any non-representable graphs that do not satisfy the conditions of Theorem 10? In particular, are there any triangle-free or* 3*-chromatic non-representable graphs?*

Most likely, answers to the questions above are positive. A good candidate for being a counterexample is the famous Petersen's graph. It is 3-chromatic, triangle-free and so resistable to all our attempts to represent it, that we have reasons to formulate the following

Conjecture 1 *The Petersen's graph is non-representable.*

Another problem, that may be closely related to Problem 1 is the algorithmic complexity of recognition whether a given graph is representable or not.

Problem 2 *Is it an NP-hard problem to find out whether a graph is representable or not?*

Really, it is not even proven that this problem is in NP (it is necessary to show that any representable graph on n vertices may be represented by a word whose length is bounded by a polynomial $p(n)$).

Finally, we remind the main (enumerative) problem from application point of view:

Problem 3 *How many representable graphs on* n *vertices are there? Can one provide good lower and/or upper bounds for this number?*

Note, however, that almost all graphs are non-representable since almost all graphs contain the wheel W_5 (the leftmost graph in Figure 1) as an induced subgraph.

References

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