

On graphs with representation number 3

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Abstract

A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if (x, y) is an edge in E . A graph is word-representable if and only if it is k -word-representable for some k , that is, if there exists a word containing k copies of each letter that represents the graph. Also, being k -word-representable implies being $(k + 1)$ -word-representable. The minimum k such that a word-representable graph is k -word-representable, is called graph's representation number.

Graphs with representation number 1 are complete graphs, while graphs with representation number 2 are circle graphs. The only fact known before this paper on the class of graphs with representation number 3, denoted by \mathcal{R}_3 , is that the Petersen graph and triangular prism belong to this class. In this paper, we show that any prism belongs to \mathcal{R}_3 , and that two particular operations of extending graphs preserve the property of being in \mathcal{R}_3 . Further, we show that \mathcal{R}_3 is not included in a class of c -colorable graphs for a constant c . To this end, we extend three known results related to operations on graphs.

We also show that ladder graphs used in the study of prisms are 2-word-representable, and thus each ladder graph is a circle graph. Finally, we discuss k -word-representing comparability graphs via consideration of crown graphs, where we state some problems for further research.

Keywords: word-representable graph, representation number, prism, ladder graph, circle graph, crown graph, comparability graph

1 Introduction

A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet $V = V(G)$ such that letters x and y alternate in w if and only if (x, y) is an edge in $E = E(G)$. It follows from definitions that word-representable graphs are a *hereditary class* of graphs. A comprehensive introduction to the theory of word-representable graphs is given in [3].

A graph is word-representable if and only if it is k -word-representable for some k , that is, if there exists a word containing k copies of each letter that represents the graph (see Theorem 3). By Proposition 4, being k -word-representable implies being $(k + 1)$ -word-representable. The

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minimum k such that a word-representable graph is k -word-representable, is called *graph's representation number*. We let \mathcal{R}_k denote the class of graphs having representation number k .

\mathcal{R}_1 is easy to see to be the class of complete graphs (also known as cliques), while \mathcal{R}_2 is the class of circle graphs (see Theorem 10). The only fact known before this paper on \mathcal{R}_3 was that the Petersen graph (to the right in Figure 1) and the triangular prism (to the left in Figure 5) belong to this class. Theorem 19 extends our knowledge on \mathcal{R}_3 , in particular, showing that *all* prisms belong to this class (also, see Theorem 18).

In Section 4, we revise connecting two word-representable graphs, say G_1 and G_2 , by an edge and gluing these graphs in a vertex, originally studied in [4]. One can use a graph orientation argument involving Theorem 7, or results in [4], to show that the resulting graph G , in both cases, is word-representable. However, these do not answer directly the following question: If G_1 is k_1 -word-representable, G_2 is k_2 -word-representable, and G is k -word-representable (such a k must exist by Theorem 3) then what can be said about k ? Theorem 22 answers the question.

Further, in Section 5 we revise replacing a vertex in a graph with a module, briefly considered in [1]. Theorem 23 in that section is an extended version of an observation made in [1]. In Section 6, providing two arguments based on Theorems 22 and 23, we show that \mathcal{R}_3 is not included in a class of c -colorable graphs for a constant c .

In Section 7 we show that ladder graphs used in the study of prisms in [4] are 2-word-representable, and thus each ladder graph is a circle graph. Finally, in Section 8 we discuss k -word-representing comparability graphs via consideration of crown graphs, where we state some problems for further research.

2 Preliminaries

Suppose that w is a word and x and y are two distinct letters in w . We say that x and y *alternate* in w if after deleting in w all letters but the copies of x and y we either obtain a word $xyxy \cdots$ (of even or odd length) or a word $yxyx \cdots$ (of even or odd length). If x and y do not alternate in w , we say that these letters are *non-alternating* in w . For example, if $w = 31341232$ then the letters 1 and 3 are alternating in w because removing all other letters we obtain 31313, while 3 and 4 are non-alternating because removing all other letters from w we have 3343.

A simple graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if $(x, y) \in E$ for each $x \neq y$. We say that w *represents* G , and w is called a *word-representant* for G . The graphs in Figure 1 are word-representable. Indeed, for example, 1213423 is a word-representant for M , 1234 is a word-representant for K_4 , and a word-representant for the Petersen graph is

$$1387296(10)7493541283(10)7685(10)194562.$$

In what follows, we will need the following two propositions that are easy to see from definitions.

Proposition 1. ([4]) *Let $w = w_1xw_2xw_3$ be a word representing a graph G , where w_1 , w_2 and w_3 are possibly empty words, and w_2 contains no x . Let X be the set of all letters that appear only once in w_2 . Then possible candidates for x to be adjacent to in G are the letters in X .*

Proposition 2. *If w represents G , then the reverse $r(w)$, which is writing the letters in w in the reverse order, also represents G .*

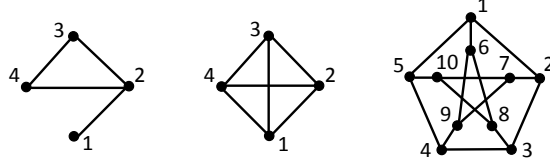


Figure 1: Three word-representable graphs M (left), the complete graph K_4 (middle), and the Petersen graph (right).

2.1 k -word-representable graphs

A word is k -uniform if each letter in it appears exactly k times. In particular, 1-uniform words are permutations. A graph is k -word-representable if there exists a k -uniform word representing it. We say that the word k -represents the graph.

The following theorem is of special importance in the theory of word-representable graphs.

Theorem 3. ([4]) *A graph G is word-representable if and only if it is k -word-representable for some k .*

In what follows, we will also use the following two propositions.

Proposition 4. ([4]) *A k -word-representable graph G is also $(k + 1)$ -word-representable. In particular, each word-representable graph has infinitely many word-representants representing it since for every $\ell > k$, a k -word-representable graph is also ℓ -word-representable.*

Proposition 5. ([4]) *Let $w = uv$ be a k -uniform word representing a graph G , where u and v are two, possibly empty, words. Then the word $w' = vu$ also represents G .*

2.2 Permutationally representable graphs

A graph G with the vertex set $V = \{1, \dots, n\}$ is *permutationally representable* if it can be represented by a word of the form $p_1 \cdots p_k$, where p_i is a permutation of V for $1 \leq i \leq k$. For example, a complete graph K_n is permutationally representable for any n . Indeed, take any permutation of $\{1, \dots, n\}$ and repeat it as many times (maybe none) as desired. For another example, the path $1 - 2 - 3$ is also permutationally representable, and one such representation is 213231, while some other such representations can be obtained from it by adjoining any number of permutations 213 and 231 to it. If G can be represented permutationally involving k permutations, we say that G is *permutationally k -representable*.

An orientation of graph's edges is called *transitive* if having (directed) edges $x \rightarrow y$ and $y \rightarrow z$ for some vertices x, y, z implies having the edge $x \rightarrow z$. A graph is a *comparability graph* if it accepts a transitive orientation.

Theorem 6. ([5]) *A graph is permutationally representable if and only if it is a comparability graph.*

Comparability graphs correspond to partially ordered sets (posets), and the question whether such a graph can be permutationally k -represented is equivalent to the question whether the respective poset can be represented as an intersection of k linear orders. The minimum such

k for a poset is called the *poset dimension*. We call *k-comparability graphs* those graphs that correspond to posets having dimension k , that is, those graphs that are permutationally k -representable but not permutationally $(k - 1)$ -representable.

2.3 Semi-transitive orientations

A graph $G = (V, E)$ is *semi-transitive* if it admits an acyclic orientation such that for any directed path $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$ with $v_i \in V$ for all i , $1 \leq i \leq k$, either

- there is no edge $v_1 \rightarrow v_k$, or
- the edge $v_1 \rightarrow v_k$ is present and there are edges $v_i \rightarrow v_j$ for all $1 \leq i < j \leq k$. That is, in this case, the (acyclic) subgraph induced by the vertices v_1, \dots, v_k is transitive (with the unique source v_1 and the unique sink v_k).

We call such an orientation *semi-transitive orientation*. For example, the orientation of the graph in Figure 2 is semi-transitive, and thus the underlying (non-directed) graph is semi-transitive.

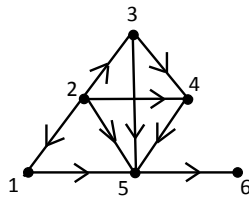


Figure 2: An example of a semi-transitive orientation.

Clearly, all transitive (that is, comparability) graphs are semi-transitive, and thus semi-transitive orientations are a generalization of transitive orientations.

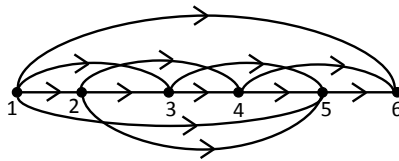


Figure 3: An example of a shortcut.

Semi-transitive orientations are defined in [2] in terms of *shortcuts* as follows. A *semi-cycle* is the directed acyclic graph obtained by reversing the direction of one edge of a directed cycle. An acyclic digraph is a shortcut if it is induced by the vertices of a semi-cycle and contains a pair of non-adjacent vertices. Thus, a digraph on the vertex set $\{v_1, \dots, v_k\}$ is a shortcut if it contains a directed path $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_k$, the edge $v_1 \rightarrow v_k$, and it is missing an edge $v_i \rightarrow v_j$ for some $1 \leq i < j \leq k$; in particular, we must have $k \geq 4$, so that any shortcut is on at least four vertices. See Figure 3 for an example of a shortcut (there, the edges $1 \rightarrow 4$, $2 \rightarrow 6$, and $3 \rightarrow 6$ are missing). An orientation of a graph is semi-transitive, if it is acyclic and contains no shortcuts. Clearly, this definition is just another way to introduce the notion of semi-transitive orientations presented above.

The following theorem is a useful characterization of word-representable graphs that allows answering questions on word-representability in terms of graph orientations.

Theorem 7. ([2]) *A graph G is word-representable if and only if it is semi-transitive (that is, it accepts a semi-transitive orientation).*

A direct corollary to the last theorem is the following statement.

Theorem 8. ([2]) *3-colorable graphs are word-representable.*

2.4 Graph's representation number

The following statement is easy to see.

Theorem 9. *A graph G is in \mathcal{R}_1 if and only if $G = K_n$, the complete graph on n vertices, for some n .*

We also have a characterization of graphs in \mathcal{R}_2 . To state it we need the following definition: A *circle graph* is an undirected graph whose vertices can be associated with chords of a circle such that two vertices are adjacent if and only if the corresponding chords cross each other.

Theorem 10. ([2]) *For a graph G different from a complete graph, G is in \mathcal{R}_2 if and only if G is a circle graph.*

The following proposition is not difficult to see from definitions.

Proposition 11. *If $\mathcal{R}(G) = k$ and G' is an induced subgraph of G then $\mathcal{R}(G') \leq k$.*

Proof. Indeed, using the hereditary nature of word-representable graphs, if representing G' would require more than k copies of each letter, then representing G would obviously require more than k copies of each letter. \square

Theorem 12. ([2]) *Each word-representable graph on n vertices is n -word-representable.*

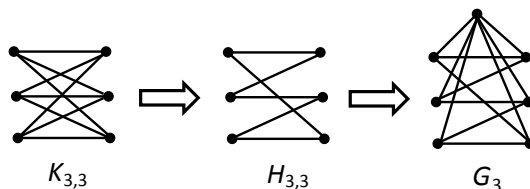


Figure 4: Graphs involved in defining the graph G_3 .

It turns out that there are graphs on n vertices with representation number of $\lfloor \frac{n}{2} \rfloor$, matching the upper bound in Theorem 12 within a factor of 2. A *crown graph* $H_{k,k}$ is a graph obtained from the complete bipartite graph $K_{k,k}$ by removing a perfect matching (see Figure 10 for examples of crown graphs). Then G_k is the graph obtained from a crown graph $H_{k,k}$ by adding an all-adjacent vertex. See Figure 4 for graphs involved in defining G_3 , where note that all choices in defining $H_{3,3}$ lead to isomorphic graphs.

Theorem 13. ([2]) *The graph G_k on $2k + 1$ vertices belongs to \mathcal{R}_k .*

2.5 Two operations to extend a graph

The following theorem gives a useful tool to construct 3-word-representable graphs, that is, graphs with representation number at most 3.

Theorem 14. ([4]) *Let $G = (V, E)$ be a 3-word-representable graph and $x, y \in V$. Denote by H the graph obtained from G by adding to it a path of length at least 3 connecting x and y . Then H is also 3-word-representable.*

We also have the following proposition.

Proposition 15. *Let $G \in \mathcal{R}_k$, where $k \geq 2$, and $x \in V(G)$. Also, let G' be the graph obtained from G by adding an edge (x, y) , where $y \notin V(G)$. Then $G' \in \mathcal{R}_k$.*

Proof. Suppose that G is k -represented by a word $w_0 x w_1 x w_2 \cdots x w_{k-1} x w_k$, where for $0 \leq i \leq k$, w_i is a word not containing x . Then it is not difficult to check that the word

$$w_0 y x y w_1 x w_2 y x w_3 y x w_4 \cdots y x w_{k-1} y x w_k$$

k -represents G' (in particular, the vertex x is the only neighbor of y). Finally, if G' could be $(k-1)$ -represented by some word, we would remove from that word the letter y to obtain a $(k-1)$ -representation of G , which is impossible. So, $G' \in \mathcal{R}_k$. \square

3 Prisms and \mathcal{R}_3

A *prism* Pr_n is a graph consisting of two cycles $12 \cdots n$ and $1'2' \cdots n'$, where $n \geq 3$, connected by the edges (i, i') for $i = 1, 2, \dots, n$. In particular, the 3-dimensional cube is a prism. Examples of prisms are given in Figure 5. The leftmost prism there is called the *triangular prism*. The middle prism is the 3-dimensional cube.

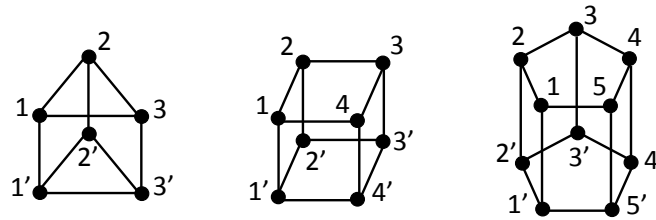


Figure 5: Examples of prisms.

Theorem 16. ([4]) *Every prism Pr_n is 3-word-representable.*

Theorem 17. ([4]) *The triangular prism Pr_3 is not 2-word-representable, and thus, by Theorem 16, $\mathcal{R}(Pr_3) = 3$.*

Our next theorem extends Theorem 17 by showing that *any* prism belongs to \mathcal{R}_3 .

Theorem 18. *For $n \geq 4$, Pr_n is not 2-word-representable, and thus, by Theorem 16, $\mathcal{R}(Pr_n) = 3$.*

Proof. Suppose that Pr_n can be 2-represented by a word w for $n \geq 4$. It is not difficult to see that there must exist a letter x in w such that no letter occurs twice between the two copies of x . By Proposition 1, there are exactly three letters between the copies of x . Using symmetry, Proposition 5 and Proposition 2, we only need to consider two cases (the second one is unnecessary in the case of $n = 4$ because of symmetry) where we took into account that the vertices $1'$, 2 and n form an independent set:

- w is of the form $11'2n1 \cdots n \cdots 2 \cdots 1' \cdots$. Since $(n, n') \in E(\text{Pr}_n)$ and $(1', n') \in E(\text{Pr}_n)$, we can refine the structure of w as follows

$$w = 11'2n1 \cdots n' \cdots n \cdots 2 \cdots 1' \cdots n' \cdots .$$

However, 2 and n' alternate in w contradicting to the fact that $(2, n') \notin E(\text{Pr}_n)$.

- w is of the form $121'n1 \cdots n \cdots 1' \cdots 2 \cdots$. In this case, we will refine the structure of w in two different ways and then will merge these refinements:

- Since $(2, 2') \in E(\text{Pr}_n)$, $(1', 2') \in E(\text{Pr}_n)$ and $(2', n) \notin E(\text{Pr}_n)$, w must be of the form

$$w = 121'n1 \cdots n \cdots 2' \cdots 1' \cdots 2 \cdots 2' \cdots .$$

- Since $(n, n') \in E(\text{Pr}_n)$, $(1', n') \in E(\text{Pr}_n)$ and $(2, n') \notin E(\text{Pr}_n)$, w must be of the form

$$w = 121'n1 \cdots n' \cdots n \cdots 1' \cdots n' \cdots 2 \cdots$$

Merging the refinements, we see that w must be of the form

$$w = 121'n1 \cdots n' \cdots n \cdots 2' \cdots 1' \cdots n' \cdots 2 \cdots 2' \cdots .$$

However, we see that the letters $2'$ and n' alternate in w contradicting to the fact that $(2', n') \notin E(\text{Pr}_n)$. □

Applying Proposition 15 as many times as necessary, we see that if $G \in \mathcal{R}_3$ then a graph obtained from G by attaching simple paths of any lengths to vertices in G belongs to \mathcal{R}_3 ; call this way to extend graphs “operation 1”. Also, by Theorem 14, we can add simple paths of length at least 3 connecting any pair of vertices in $G \in \mathcal{R}_3$ and still obtain a graph in \mathcal{R}_3 (if a 2-word-representable graph would be obtained, we would have a contradiction with $G \in \mathcal{R}_3$); call this way to extend graphs “operation 2”. The only known to us for the moment graphs in \mathcal{R}_3 are recorded in the following statement whose truth follows from considerations above.

Theorem 19. \mathcal{R}_3 contains the Petersen graph, prisms, and any other graph obtained from these by applying operations 1 and 2 arbitrary number of times in any order.

4 Connecting two graphs by an edge and gluing two graphs in a vertex

The operations of connecting two graphs, G_1 and G_2 , by an edge and gluing these graphs in a vertex are presented schematically in Figure 6. It follows directly from Theorem 7 that if both G_1 and G_2 are word-representable then the resulting graphs will be word-representable too, while if at least one of G_1 or G_2 is non-word-representable then the resulting graphs will be non-word-representable. Indeed, if G_1 and G_2 are oriented semi-transitively, then orienting the edge (x, y) in either direction will not give a chance for the resulting graph to have a shortcut (defined in Subsection 2.3) thus resulting in a semi-transitively oriented graph; similarly, no shortcut is possible when semi-transitively oriented G_1 and G_2 are glued in a vertex z .

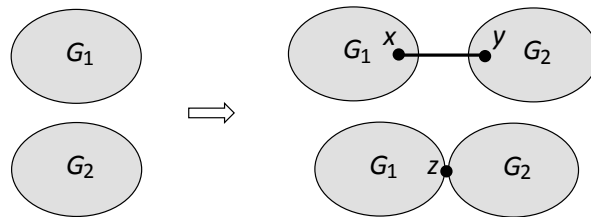


Figure 6: Connecting graphs by an edge and gluing graphs in a vertex.

While the arguments above involving orientations answer the question on word-representability of connecting graphs by an edge or gluing graphs in a vertex, they do not allow to answer the following question: If G_1 is k_1 -word-representable, G_2 is k_2 -word-representable, and G is k -word-representable (such a k must exist by Theorem 3) then what can be said about k ? Theorem 22 below, that is based on Theorems 20 and 21, answers this question.

Theorem 20. ([4]) For $k \geq 2$, let w_1 and w_2 be k -uniform words representing graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, where V_1 and V_2 are disjoint. Suppose that $x \in V_1$ and $y \in V_2$. Let H_1 be the graph $(V_1 \cup V_2, E_1 \cup E_2 \cup \{(x, y)\})$. Then H_1 is k -word-representable.

Theorem 21. ([4]) For $k \geq 2$, let w_1 and w_2 be k -uniform words representing graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively, where V_1 and V_2 are disjoint. Suppose that $x \in V_1$ and $y \in V_2$. Let H_2 be the graph obtained from G_1 and G_2 by identifying x and y into a new vertex z . Then H_2 is k -word-representable.

Theorem 22. Suppose that for graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $\mathcal{R}(G_1) = k_1$ and $\mathcal{R}(G_2) = k_2$, $x \in V_1$, $y \in V_2$ and $k = \max(k_1, k_2)$. Also, let the graph G' be obtained by connecting G_1 and G_2 by the edge (x, y) , and the graph G'' be obtained from G_1 and G_2 by identifying the vertices x and y into a single vertex z . The following holds.

1. If $|V_1| = |V_2| = 1$ then both G' and G'' are cliques and thus $k_1 = k_2 = 1$. In this case, $\mathcal{R}(G') = \mathcal{R}(G'') = 1$.
2. If $\min(|V_1|, |V_2|) = 1$ but $\max(|V_1|, |V_2|) > 1$ then $\mathcal{R}(G'') = k$ and $\mathcal{R}(G') = \max(k, 2)$.
3. If $\min(|V_1|, |V_2|) > 1$ then $\mathcal{R}(G') = \mathcal{R}(G'') = \max(k, 2)$.

Proof. The first part of the statement is easy to see since both a single vertex (G'') and the one edge graph (G') are 1-word-representable by Theorem 9.

For part 2, without loss of generality, $|V_1| = 1$ (that is, $V_1 = \{x\}$) and thus $G'' = G_2$ leading to $\mathcal{R}(G'') = k$. On the other hand, G' is not a clique and thus $\mathcal{R}(G') \geq 2$. If $k_2 = 1$ then G_2 can be represented by the permutation $yy_1 \cdots y_{|V_2|-1}$ for $y, y_i \in V_2$ and thus G' can be represented by $xyxy_1 \cdots y_{|V_2|-1}yy_1 \cdots y_{|V_2|-1}$ leading to $\mathcal{R}(G') = 2$. However, if $k_2 \geq 2$, so that $k = k_2$, we can take any k -word-representation of G_2 and replace in it every other occurrence of the letter y by xyx to obtain a k -word-representation of G' . Thus, $\mathcal{R}(G') = k$ because if it would be less than k , we would have $\mathcal{R}(G_2) < k$ by Proposition 11, a contradiction.

For part 3, neither G' nor G'' is a clique and thus $\mathcal{R}(G'), \mathcal{R}(G'') \geq 2$. By Proposition 4, both G_1 and G_2 are k -word-representable. If $k \geq 2$ then by Theorems 20 and 21 both G' and G'' are k -word-representable leading to $\mathcal{R}(G') = \mathcal{R}(G'') = k$ since if $\mathcal{R}(G') < k$ or $\mathcal{R}(G'') < k$ we would obtain a contradiction either with $\mathcal{R}(G_1) = k_1$ or with $\mathcal{R}(G_2) = k_2$ by Proposition 11. Finally, if $k = 1$ then G_1 and G_2 must be cliques that can be represented by permutations $x_1 \cdots x_{|V_1|-1}x$ and $yy_1 \cdots y_{|V_2|-1}$, respectively, for $x, x_i \in V_1$ and $y, y_i \in V_2$. Then the words

$$x_1 \cdots x_{|V_1|-1}xx_1 \cdots x_{|V_1|-1}yxy_1 \cdots y_{|V_2|-1}yy_1 \cdots y_{|V_2|-1}$$

and

$$x_1 \cdots x_{|V_1|-1}zx_1 \cdots x_{|V_1|-1}y_1 \cdots y_{|V_2|-1}zy_1 \cdots y_{|V_2|-1}$$

2-word-represent the graphs G' and G'' , respectively, and thus $\mathcal{R}(G') = \mathcal{R}(G'') = 2$. \square

5 Replacing a vertex in a graph with a module

A subset X of the set of vertices V of a graph G is a *module* if all members of X have the same set of neighbors among vertices not in X (that is, among vertices in $V \setminus X$). For example, Figure 7 shows replacing the vertex 1 in the triangular prism by the module K_3 formed by the vertices a, b and c .

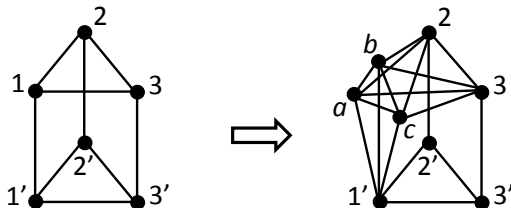


Figure 7: Replacing a vertex by a module.

The following theorem is an extended version of an observation made in [1].

Theorem 23. *Suppose that G is a word-representable graph and $x \in V(G)$. Let G' be obtained from G by replacing x with a module M , where M is any comparability graph (in particular, any clique). Then G' is also word-representable. Moreover, if $\mathcal{R}(G) = k_1$ and $\mathcal{R}(M) = k_2$ then $\mathcal{R}(G') = k$, where $k = \max\{k_1, k_2\}$.*

Proof. By Theorem 6, M can be represented by a word $p_1 \cdots p_{k_2}$, where p_i is a permutation (of length equal to the number of vertices in M) for $1 \leq i \leq k_2$. If $k_2 < k$, we can adjoin to the representation of M any number of copies of permutations in the set $\Pi = \{p_1, \dots, p_{k_2}\}$ to obtain a k -representation $p = p_1 \cdots p_k$ of M , where $p_i \in \Pi$ (indeed, no alternation properties will be changed while adjoining such extra permutations).

Using Proposition 4, if necessary (in case when $k_1 < k$), we can assume that G can be k -represented by a word $w = w_0 x w_1 x w_2 \cdots x w_k$ for some words w_i not containing x for all $0 \leq i \leq k$. But then the word $w' = w_0 p_1 w_1 p_2 w_2 \cdots p_k w_k$ k -represents G' . Indeed, it is easy to see that the pairs of letters in w' from $V(M)$ have the right alternation properties, as do the pairs of letters from $V(G) \setminus \{x\}$ in w' . On the other hand, it is not difficult to see that if $y \in V(M)$ and $z \in V(G) \setminus \{x\}$ then $(y, z) \in E(G')$ if and only if $(x, z) \in E(G)$.

If G' would be $(k - 1)$ -word-representable, we would either obtain a contradiction with $\mathcal{R}(G) = k_1$ (after replacing each p_i in w' with x) or with $\mathcal{R}(M) = k_2$ (after removing all letters in w' that are not in M). Thus, $\mathcal{R}(G') = k$. \square

6 Graphs in \mathcal{R}_3 and c -colorable graphs

Theorems 9 and 10 show that there exists no constant c such that all 1- or 2-word-representable graphs are c -colorable. Indeed, K_n is n -colorable, while circle graphs formed by K_{n-1} and an isolated vertex (which are not 1-word-representable) are $(n - 1)$ -colorable. On the other hand, known to us graphs G_k (considered in Theorem 13) that require many copies of each letter to be represented, are 3-colorable for any $k \geq 2$ (because G_k is a bipartite graph with an all-adjacent vertex). Thus, 3-word-representable graphs do not contain a class of c -colorable graphs for some constant $c \geq 3$ (this claim follows from the fact that G_k , being 3-colorable, is not 3-word-representable for $k \geq 4$).

A natural question to ask here is: Is \mathcal{R}_3 properly included in a class of c -colorable graphs for a constant c ? The following theorem shows that this is not the case.

Theorem 24. *The class \mathcal{R}_3 is not included in a class of c -colorable graphs for some constant c .*

Proof. Suppose that any graph in class \mathcal{R}_3 is c -colorable for some constant c . We can assume that $c \geq 3$ since G_3 being 3-colorable belongs to \mathcal{R}_3 by Theorem 13. Consider the triangular prism Pr_3 to the left in Figure 7, which is also 3-colorable and, by Theorem 17, belongs to \mathcal{R}_3 . Replace the vertex 1 in Pr_3 with a module K_{c+1} , the complete graph on $c + 1$ vertices (see Section 5 for the notion of a module) as shown for the case of $c = 2$ in Figure 7. Denote the obtained graph by Pr'_3 . Since $\mathcal{R}(K_{c+1}) = 1$, by Theorem 23, $Pr'_3 \in \mathcal{R}_3$. However, Pr'_3 is not c -colorable since it contains a clique of size $c + 1$ (Pr'_3 is $(c + 1)$ -colorable). We obtain a contradiction with our assumption. \square

We note that an alternative way to obtain a contradiction in the proof of Theorem 24 is to consider the triangular prism Pr_3 (which is in \mathcal{R}_3), the complete graph K_{c+1} (which is in \mathcal{R}_1), and either to connect these graphs by an edge, or glue these graphs in a vertex. Then by Theorem 22, the obtained graph will be in \mathcal{R}_3 , but it is $(c + 1)$ -colorable.

From considerations in this section it follows that, in particular, the classes \mathcal{R}_3 and 3-colorable graphs (that are word-representable by Theorem 8) are not comparable in the sense

that none of these classes is included in the other one. However, this section does not answer the following question.

Problem 1. *Can each bipartite (that is, 2-colorable) graph be 3-word-represented? Namely, does each bipartite graph belong to the union of the sets $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$?*

We suspect the answer to the question in Problem 1 to be negative. A good candidate for a counterexample should be the crown graph $H_{k,k}$ for $k \geq 5$; see Section 8 and, in particular, Problem 3.

7 Ladder graphs

The ladder graph L_n with $2n$ vertices and $3n - 2$ edges is presented in Figure 8.

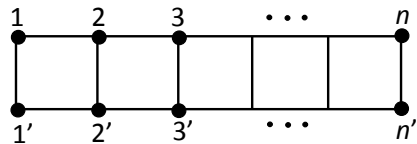


Figure 8: The ladder graph L_n .

It follows from the proof of Theorem 16 in [4] that L_n is 3-word-representable. In the following theorem, we will show that L_n is actually 2-word-representable for $n \geq 2$ (it is clearly 1-word-representable for $n = 1$).

Theorem 25. *For $n \geq 2$, $\mathcal{R}(L_n) = 2$. Thus, by Theorem 10, any ladder graph is a circle graph.*

Proof. We prove the statement by induction on n . L_1 can be 2-represented by the word $w_1 = 11'11'$ that has the factor $1'1$. Substituting the factor $1'1$ in w_1 by $2'1'22'12$ and reversing the entire word, we obtain the word $w_2 = 1'212'21'2'1$ that contains the factor $2'2$. It is straightforward to check that w_2 represents L_2 since one only needs to check the alternation properties of the just added letters 2 and $2'$.

More generally, given a 2-representation w_i of L_i containing the factor $i'i$, we substitute $i'i$ in w_i by $(i+1)'i'(i+1)(i+1)'i(i+1)$ and reverse the entire word to obtain the word w_{i+1} containing the factor $(i+1)'(i+1)$. It is straightforward to check that w_{i+1} represents L_{i+1} since the only thing that needs to be checked is the right alternation properties of the just added letters $i+1$ and $(i+1)'$. We are done. \square

In Table 1, we record 2-representations of the ladder graph L_n for $n = 1, \dots, 5$. The factors $n'n$ are indicated in bold.

Of course, an alternative proof of Theorem 25 would be in representing vertices in L_n by properly overlapping chords on a circle. However, we find our proof of that theorem a bit easier to follow/record.

Remark 26. *The thoughtful Reader would notice what seems to be an inconsistency: our argument in the proof of Theorem 18 seems to show that the graph in Figure 9 is not 2-word-representable, while by Theorem 25 we see that this graph is 2-word-representable by*

n	2-representation of the ladder graph L_n
1	11'11'
2	1'212'21'2'1
3	12'1'323'32'3'121'
4	1'213'2'434'43'4'231'2'1
5	12'1'324'3'545'54'5'342'3'121'

Table 1: 2-representation of the ladder graph L_n for $n = 1, \dots, 5$.

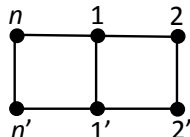


Figure 9: The ladder graph on 6 vertices.

$n1'n'212'21'2'n1n'$ (we renamed the labels in L_3 respectively, and used the third line in Table 1). The reason for possible confusion is that we used symmetry in the proof of Theorem 18 to assume that there are exactly three letters between the 1s; such an assumption cannot be made while dealing with the graph in Figure 9 because 1 is an “internal” vertex there, while there are “external” vertices as well, namely 2, 2', n, n'.

8 Crown graphs

Recall definition of the crown graph $H_{k,k}$ in Section 2.4 and see Figure 10 for a few small such graphs. It is a well-known fact that the dimension of the poset corresponding to $H_{k,k}$ is k for $k \geq 2$, and thus $H_{k,k}$ is a k -comparability graph (it is permutationally k -representable but not permutationally $(k - 1)$ -representable). In [2], the following way to represent permutationally $H_{k,k}$ was suggested. Concatenate the permutation $12 \cdots (k - 1)k'k(k - 1)' \cdots 2'1'$ together with all permutations obtained from this by simultaneous exchange of k and k' with m and m' , respectively, for $m = 1, \dots, k - 1$. See Table 2 for permutationally k -representation of $H_{k,k}$ for $k = 1, 2, 3, 4$.

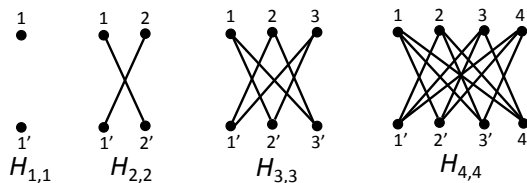


Figure 10: The crown graph $H_{k,k}$ for $k = 1, 2, 3, 4$.

It is not difficult to see by induction on the number of vertices, that each tree is 2-word-representable (just add a new leaf y to a vertex x in a tree T , and substitute an x in the word representing T with xy). Based on this, one can show that any cycle graph is 2-word-representable (one first represents a path, which is a tree, and then adds one more edge by

k	permutationally k -representation of the crown graph $H_{k,k}$
1	11'1'1
2	12'21'21'12'
3	123'32'1'132'23'1'231'13'2'
4	1234'43'2'1'1243'34'2'1'1342'24'3'1'2341'14'3'2'

Table 2: Permutationally k -representation of the crown graph $H_{k,k}$ for $k = 1, 2, 3, 4$.

swapping a certain pair of consecutive letters in the word representing the path). 2-representing trees and cycles is discussed in detail in [3].

Consider the cycle graph C_5 on five edges. C_5 is in \mathcal{R}_2 (it is 2-word-representable, but not 1-word-representable), but it is not a comparability graph and thus, by Theorem 6, C_5 is not permutationally representable. On the other hand, the crown graphs $H_{1,1}$ and $H_{2,2}$, being 2-comparability graphs, belong to \mathcal{R}_2 . Also, $H_{3,3}$, being a 3-comparability graph, belongs to \mathcal{R}_2 , since $H_{3,3}$ is the cycle graph C_6 . Moreover, $H_{4,4}$, being a 4-comparability graph, belongs to \mathcal{R}_4 , which follows from the fact that $H_{4,4}$ is the prism Pr_4 (the 3-dimensional cube) and Theorem 18 can be applied.

Crown graphs, being bipartite graphs, and thus comparability graphs, provide an interesting case study of relations between k -comparability graphs and k -word-representable graphs. While each k -comparability graph is necessarily k -word-representable, in some cases such a graph is also $(k - 1)$ -word-representable, and, in fact, it is in \mathcal{R}_{k-1} in the known to us situations. Thus, it seems like giving up permutational representability, we should be able to come up with a shorter representation of a given comparability graph. However, we do not know whether this is essentially always the case (except for some particular cases like the graphs $H_{1,1}$ and $H_{2,2}$). Thus we state the following open problem.

Problem 2. *Characterize k -comparability graphs that belong to $\mathcal{R}_{k-\ell}$ for a fixed ℓ . In particular, characterize those k -comparability graphs that belong to \mathcal{R}_{k-1} . Is the set of k -comparability graphs that belong to $\mathcal{R}_{k-\ell}$ (non-)empty for a fixed $\ell \geq 2$?*

A step towards solving Problem 2 could be first understanding crown graphs and solving the following problem.

Problem 3. *Characterize those $H_{k,k}$ that belong to $\mathcal{R}_{k-\ell}$ for a fixed ℓ . In particular, characterize those $H_{k,k}$ that belong to \mathcal{R}_{k-1} . Is the set of crown graphs $H_{k,k}$ that belong to $\mathcal{R}_{k-\ell}$ (non-)empty for a fixed $\ell \geq 2$?*

We suspect that each $H_{k,k}$ belongs to \mathcal{R}_{k-1} for $k \geq 3$. Even if we would manage to prove this statement for a single $k \geq 5$, we would answer at once (negatively) the question in Problem 1. A step towards solving Problem 3 can be the following observation: Take the concatenation of the permutations representing $H_{k,k}$ discussed above and remove the leftmost $k - 1$ letters $1, 2, \dots, k - 1$ and the rightmost $k - 1$ letters $2', 3', \dots, k'$. The remaining word will still represent $H_{k,k}$, which is not difficult to see. One would then need to remove from the word two more letters, one copy of k and one copy of $1'$, then possibly do some rearrangements of the remaining letters with a hope to obtain a word that would $(k - 1)$ -word-represent $H_{k,k}$. For example, for

$k = 3$, we begin with the word representing $H_{3,3}$ in Table 2, and remove its first two, and its last two letters to obtain the following representation of $H_{3,3}$:

$$3'32'1'132'23'1'231'1.$$

Now, remove the leftmost 3 and the leftmost $1'$, make the last letter 1 to be the first letter 1 (that is, make the cyclic one-position shift to the right), and, finally, replace the six rightmost letters in the obtained word by the word obtained by listing these letters in the reverse order to obtain a 2-word-representation of $H_{3,3}$ (this is the same representation as one would obtain for the cycle C_6 if to follow the strategy described in [3]): $13'2'132'1'321'3'2$.

Unfortunately, similar steps do not work for 3-word-representing $H_{4,4}$, and it is not so clear which copies of unwanted letters 4 and $1'$ one should remove before conducting further rearrangements. In either case, even if one could demonstrate how to obtain a $(k - 1)$ -word-representation of $H_{k,k}$ for some $k \geq 5$, it is not so clear what generic argument would show that no $(k - 2)$ -word-representation exists, if this would be the case.

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