ON SQUARE-FREE PERMUTATIONS

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Abstract. A permutation is square-free if it does not contain two consecutive factors of length more than one that coincide in the reduced form (as patterns). We prove that the number of square-free permutations of length n is $n^{n(1-\varepsilon_n)}$ where $\varepsilon_n \to 0$ when $n \to \infty$.

A permutation of length n is crucial with respect to squares if it avoids squares but any extension of it to the right, to a permutation of length $n+1$, contains a square. A permutation is maximal with respect to squares if both the permutation and its reverse are crucial with respect to squares. We prove that there exist crucial permutations with respect to squares of any length at least 7, and there exist maximal permutations with respect to squares of odd lengths $8k+1, 8k+5, 8k+7$ for $k \geq 1$.

Keywords: square freeness, consecutive pattern, enumeration, crucial word, maximal word, permutation

MSC (2000): 05A15, 68R15

1. Introduction

A *square* in a word is a classical concept in combinatorics on words meaning two equal consecutive factors in the word. For example, the word 213413413 contains the square 134134, whereas the word 2141231 is square-free. It was first established by Thue [8] that there are arbitrary long square-free words over 3 (or more) letter alphabets, whereas it is easy to see that square-free words over 2 and 1 letter alphabets are of length at most 3 and 1, respectively. A question on the number of different square-free words of length n is rather complicated: for example, for 3 letter alphabets, it is shown [6, 7] that the number of such words is $3^{cn(1-\epsilon_n)}$ where $1.30173\ldots < c < 1.30178\ldots$ and $\epsilon_n \to 0$ when $n \to \infty$ (compare this result with the first known lower bound of $2^{n/21}$ given by Brandenburg [1] for the number in question).

Another direction in study of square-free words is considering so called *crucial* and *maximal* square-free words. A word X is *crucial with respect to squares* if X is square-free, but Xx contains a square for any letter of the alphabet in question, that is, if X being square-free is not *extendable* to the right. A word is *maximal with respect to squares* if *both* X and the reverse of X (the word obtained from X by reading the letters from right to left) are crucial with respect to squares. It is known [3] that the *Zimin words* (defined recursively as $X_1 = 1$ and $X_n = X_{n-1}nX_{n-1}$) give crucial words with respect to squares of minimal length, $2^n - 1$, over an n-letter alphabet. The Zimin words are easy to see to be maximal words with respect to squares as well, which clearly gives the minimal length of such maximal words.

In this paper, we extend, in a natural way, the notion of a square in a word to that in a permutation (see the definition below). The new notion is related to the Permutation Patterns Theory. In Section 2 we provide a construction of a class of arbitrary long square-free permutations and we use it in Section 3 to find asymptotically the number of square-free permutations. Moreover, in Section 4 we study *crucial* and *maximal permutations with respect to squares* which are extensions of the notions of crucial and maximal words with respect to squares. We show that there exist crucial permutations with respect to squares of any length at least 7, and there exist maximal permutations with respect to squares of odd lengths $8k+1, 8k+5, 8k+7$ for $k \geq 1$. Finally, we provide some concluding remarks in Section 5.

2. Definitions, notations and the main construction

The *reduced form* of a permutation is obtained by substituting the i-th largest entrance of it by i. For example, the reduced form of 2648 is 1324. For the last example, we also say that the permutation 2648 forms the *pattern* 1324. A permutation is *square-free* if it does not contain two consecutive factors of length more than one that are equal in the reduced form (as patterns). For example, the permutation 246153 contains the square 4615 (in the reduced form the first and the last two letters form the pattern 12), whereas the permutation 246513 is square-free.

We say that a permutation π of length n is *crucial with respect to squares* if π avoids squares, but any its extension to the right (i.e. a permutation of length $n+1$ whose all letters, but the rightmost one, form the pattern π) is not square-free. For example, the permutations 1234 and 123 are not crucial, as the former one is not square-free, whereas the second one has a square-free extension to the right, e.g., to the permutation 1243 (the first 3 letters in the last permutation form the pattern 123). As a matter of fact, the shortest crucial permutation with respect to squares is of length 7. Below, we list all the crucial permutations of length 7 (the reader can check his/her understanding of the definition by considering any of the permutations below):

2136547, 2137546, 2146537, 2147536, 2156437, 2157436, 2167435, 3146527, 3147526, 3156427, 3157426, 3167425, 3246517, 3247516, 3256417, 3257416, 3267415, 3421675, 3521674, 3621574, 3721564, 4156327, 4157326, 4167325, 4256317, 4257316, 4267315, 4356217, 4357216, 4367215, 4521673, 4531672, 4532671, 4621573, 4631572, 4632571, 4721563, 4731562, 4732561, 5167324, 5267314, 5367214, 5467213, 5621473, 5631472, 5632471, 5641372, 5642371, 5721463, 5731462, 5732461, 5741362, 5742361, 6721453, 6731452, 6732451, 6741352, 6742351, 6751342, 6752341.

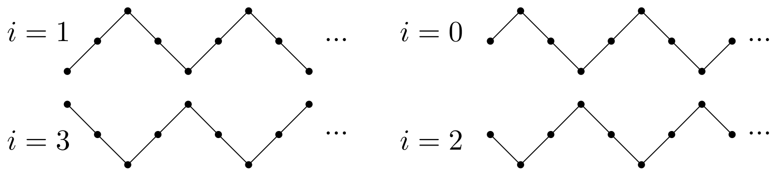
A permutation is called *maximal with respect to squares* if *both* the permutation and its reverse are crucial with respect to squares.

It is easy to see that in order to avoid squares of length 2, a permutation π must have an index $i \in \{0, 1, 2, 3\}$ so that for every non-negative integer

t the inequalities

$$\pi_{i+4t} \leq \pi_{i+4t\pm 1} \text{ and } \pi_{i+4t+2} \geq \pi_{i+4t+2\pm 1} \tag{1}$$

hold. Schematically, the four possible kinds of permutations (according to the choice of i) can be shown as follows:



Here a dot represents an element and the order of elements represented by two non-consecutive dots is irrelevant, whereas each pair of consecutive dots is comparable (a lower dot represents a smaller number).

If π is a permutation satisfying (1) then we say that the elements with the indices $4t+i$ form the *lower* level, the elements with the indices $4t+i\pm 1$ form the *medium* level, and the elements with the indices $4t+i+2$ form the *upper* level. For example, the square-free permutation 1574236 has 1 and 2 on the lower level, 7 and 6 on the upper level, and 3, 4, and 5 on the medium level. We denote the permutation induced by the medium level of the permutation π by π'.

The key thing in what follows is the following construction of a square-free permutation of length n. Choose $i \in \{0, 1, 2, 3\}$ and let $k \approx \lfloor n/4 \rfloor, \ell \approx \lfloor 3n/4 \rfloor$ (the exact values of k and ℓ depend on the parities of i and n). Take any square-free permutation π' over the elements $k, k+1, \ldots, \ell$, place it on the medium level of the permutation π to be created. Afterwards, fill in the upper (respectively lower) level with an arbitrary permutation over the elements $\ell+1, \ell+2, \ldots, n$ (respectively $1, 2, \ldots, k-1$) according to the choice of i and satisfying (1).

For example, using the permutation 4356 for the medium level, we can create, for instance, the following square-free permutations using the construction above: 42375168, 841375269, 14832576, etc.

Lemma 1. *The construction above is valid, that is, π does not contain any squares.*

Proof. Assume that π has a square W_1W_2 where W_1 and W_2 coincide as patterns. Since π satisfies (1), the length of $W_j, j = 1, 2$ is at least 4. In particular, these patterns must satisfy (1) and have the same i, i. e., the same lower, medium, and upper levels. Moreover, the patterns W_1' and W_2', induced by the corresponding medium levels must also coincide, contradicting the assumption that π' is square-free. □

3. The main result

Our main result is the following

Theorem 2. *The number $f(n)$ of square-free permutations of length n is $n^{n(1-\varepsilon_n)}$ where $\varepsilon_n \to 0$ when $n \to \infty$.*

Proof. As we know from Stirling's formula, $n! = (n/e)^n o(n)$ when $n \to \infty$. Therefore, $f(n) \leq n! \leq n^{n(1-\varepsilon_n)}$ for some $\varepsilon_n \to 0$ when $n \to \infty$.

To obtain a lower bound on $f(n)$ we use the construction presented in the previous section. Since we deal with asymptotic enumeration, we can assume that n is a power of 2, the medium level contains $n/2$ elements while each of the remaining two levels contains $n/4$ elements. Now, the medium level can be occupied by an arbitrary square-free permutation of length $n/2$, whereas on the lower and upper levels there can be any permutations of length $n/4$, which leads to

$$f(n) \geq ((n/2^2)!)^2 f(n/2) \geq \cdots \geq \prod_{i=1}^{\log_2 n} ((n/2^{i+1})!)^2$$

where we used the fact that $f(1) = 1$.

By Stirling's formula,

$$((n/2^{i+1})!)^2 \geq (n/(e2^{i+1}))^{2n/2^{i+1}} \geq (n/2^{i+3})^{n/2^i} = 2^{(\log_2 n - i - 3)(n/2^i)}.$$

So,

$$f(n) \geq \prod_{i=1}^{\log_2 n} 2^{((n \log_2 n)/2^i - n(i+3)/2^i)} = 2^{\sum_{i=1}^{\log_2 n} ((n \log_2 n)/2^i - n(i+3)/2^i)}.$$

Clearly, $\sum_{i=1}^{\log_2 n} (1/2^i) = (n-1)/n$ and

$$\sum_{i=1}^{\log_2 n} ((i+3)/2^i) \leq \sum_{i=0}^{\infty} ((i+3)/2^i) - 3 = 5.$$

Hence,

$$f(n) \geq 2^{n \log_2 n - \log_2 n - 5n} = 2^{(n \log_2 n - o(n \log_2 n))} = n^{n(1-\varepsilon_n)}$$

for some $\varepsilon_n \to 0$ when $n \to \infty$. □

4. Crucial and maximal permutations with respect to squares

We prove the following two theorems in Subsections 4.1 and 4.2, respectively.

Theorem 3. *There exist crucial permutations with respect to squares of any length at least 7.*

Theorem 4. *There exist maximal permutations with respect to squares of odd lengths* $8k+1, 8k+5, 8k+7$ *for* $k \geq 1$.

4.1. Crucial permutations with respect to squares.

The *complement* to a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is the permutation $c(\pi) = \pi'_1\pi'_2\cdots\pi'_n$, where $\pi'_i = n+1-\pi_i$. Clearly, if $\pi_i < \pi_j$ then $\pi'_i > \pi'_j$, so π is a square-free permutation if and only if $c(\pi)$ is a square-free permutation. Using the last observation and the fact of existence of square-free permutations of any length which is implicit in our main construction, for any $k \geq 3$, we can choose a square-free permutation $Y = y_1y_2\cdots y_{k-2}y_{k-1}y_k$ having $y_{k-2} < y_{k-1}$. Now, using our main construction and choosing k appropriately based on n, form a square-free permutation W' $n-1 \geq 6$ by placing Y on the medium level; moreover, we assume that W' ends with an element from the lower level. Let $W' = w'_1 w'_2 \cdots w'_{n-7} y_{k-2}\ell_1 y_{k-1} u y_k \ell_2$. Note that it is irrelevant for us whether W' begins with a letter from a lower, medium or upper level, so n can be any number greater than or equal to 7.

Form a permutation $W = W'n$. We claim that W is a crucial permutation with respect to squares. To prove the claim note that in the permutation $W(n+1)$, the factor $y_{k-2}\ell_1 y_{k-1} u y_k \ell_2 n(n+1)$ form a square, whereas if a letter $x \leq n$ is added to the right of W (increasing by 1 the letters of W that are greater than or equal to x) then the last four letters in the obtained word will form a square. It remains to show that W avoids squares.

Since W' is square-free by construction, we need to show that there are no squares in W involving n. Suppose XX is such a square. We consider three cases depending on length of X:

- $|X| = 2$. Cannot be the case as $u > y_k$ but $\ell_2 < n$.
- $|X| = 3$. Cannot be the case as $\ell_1 < y_{k-1}$ but $y_k > \ell_2$.
- $|X| \geq 4$. Since n is the largest letter in W, the last (rightmost) letter of the former X must belong to the upper level. Thus, the former X ends with the pattern 123, whereas the second X ends with the pattern 213, which is impossible.

So, W is square-free and the claim is proved.

4.2. **Maximal permutations with respect to squares.** The main idea of our construction of maximal permutations is similar to that of crucial permutations. Assume that $n = 4k+1$ for some k. First show that there is a square-free permutation $Y = y_1 y_2 \cdots y_{2k-1}$ of length at least 3 in which $y_2 < y_3$ and $y_{2k-3} < y_{2k-2}$. If $k = 2t$ then using the main construction take a square-free permutation Y starting with a letter on the lower level and ending with a letter on the upper level. It is easy to verify that such permutation has length $4t-1$ and satisfies the conditions $y_2 < y_3$ and $y_{2k-3} < y_{2k-2}$. If k is odd then add to the both sides of the permutation above a letter on the middle level (in such a way that the permutation would remain square-free; it also can be done by the main construction). Then we obtain a permutation of length $4t+1$ satisfying the necessary conditions.

Now using the main construction form a square-free permutation W' by placing Y on the medium level, starting W' with a letter on the upper level and finishing it with a letter on the lower level (W' has length $4k-1$). Increase by 1 all letters of W' and let $W = 1W'n$. Then W is a maximal permutation of length $4k+1$. Indeed, it is neither extendable to the left nor to the right without creating a square (which can be shown similarly to the crucial permutations case); W does not contain squares not involving the smallest or the largest element by construction; W does not contain squares involving just the minimum or just the maximum letter by considerations similar to the crucial permutations case; finally, W is of odd length, so it does not contain any squares involving both the minimum and the maximum elements. Thus, W is a maximal permutation with respect to squares. For example, if $Y = 123$, then W could be 174258639 (Y becomes 456 in the permutation). In fact, it is shown by exhaustive computer search that such a permutation is of minimal possible length.

A similar construction works in the case of $n = 8k + 7$. First, using the main construction take a square-free permutation Y of length $4k + 2$ beginning with a letter on the upper level and ending with a letter on the medium level. It is easy to verify that $y_2 > y_3$ and $y_{2k-3} < y_{2k-2}$. Using the main construction again form a square-free permutation W' by placing Y on the medium level, starting and finishing W' with letters on the lower level (W' has length $8k + 5$). Finally, let $W = (n - 1)W'n$. Then W is a maximal permutation of length $8k + 7$. The justification is the same as in the previous case.

Unfortunately, the constructions above do not allow us to find maximal permutations of neither length $8k + 3$ nor even length.

5. Concluding remarks

A direction of possible research is to find an upper and lower bounds for the function ϵ_n appearing in Theorem 2 or, equivalently, a lower and an upper bound for $f(n)$, respectively. For a lower bound for $f(n)$, it would be of help to discover new (rich) classes of square-free permutations (we do not expect a full characterization to be obtained). For example, one can take any permutation obtained by the construction above with the property that the minimum element m on the medium level has a neighbor a less than it; then one can substitute a with 1, m with 2, each element i, $i < a$ in the original permutation with $i + 2$ and each element i, $a < i < m$ in the original permutation with $i + 1$ keeping elements larger than m unchanged. The resulting permutation will be square-free and in most of the cases not from the class of square-free permutations considered in this paper.

For obtaining an upper bound for $f(n)$, it is helpful to realize an intimate connection to permutation pattern avoidance theory. Indeed, if a permutation satisfies (1) it must avoid 12 *consecutive* (also known as *segmented*) *patterns* of length 4: 1234, 4321, 2143, 3412, 3142, 2413, 4231, 1324, 4132, 2314, 1423, and 3241. In fact, avoiding these 12 patterns is the same as avoiding two *partially ordered consecutive patterns* $121'2'$ and $2'1'21$ where the only relations between the pattern letters are $1 < 2$ and $1' < 2'$ (see [5] to learn more on partially ordered patterns). In any case, several results and approaches are known on avoidance of consecutive patterns of length 4 (e.g., see [2, 4]), which can be used for obtaining a rough upper bound in question (instead of prohibiting 12 patterns, one can start with prohibiting 1, 2, etc, patterns out of them), and then an upper bound could be improved by including more patterns to avoid (out of the 12 patterns). It is conceivable that a direct enumeration of permutations satisfying (1) can be done without working with length 4 patterns to be avoided.

Finally, it is interesting to know if there exist any maximal permutations with respect to squares of even length or of length $8k + 3$, $k \geq 1$.

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